

# Topology of the Electroweak Vacua

Ben Gripaios\*

*Cavendish Laboratory, J.J. Thomson Ave, Cambridge, CB3 0HE, UK*

Oscar Randal-Williams†

*DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK*

In the Standard Model, the electroweak symmetry is broken by a complex,  $SU(2)$ -doublet Higgs field and the vacuum manifold  $SU(2) \times U(1)/U(1)$  has the topology of a 3-sphere. We remark that there exist alternative effective field theory descriptions that can be fully consistent with existing collider data, but in which the vacuum manifold is homeomorphic to an arbitrary non-trivial principal  $U(1)$ -bundle over a 2-sphere. These alternatives have non-trivial fundamental group and so lead to topologically-stable electroweak strings. Perhaps the most plausible alternative to  $S^3$  is the manifold  $\mathbb{R}P^3$  (with fundamental group  $\mathbb{Z}/2$ ), since it allows custodial protection of gauge boson masses and their couplings to fermions. Searches for such strings may thus be regarded as independent, and qualitatively different, precision tests of the SM, in that they are (thus far) astrophysical in nature, and test the global topology, rather than the local geometry, of the electroweak vacua.

## I. INTRODUCTION

Decades of experiment have confirmed that the weak nuclear force and the electromagnetic force are described by a gauge theory in which a group locally isomorphic to  $SU(2) \times U(1)$  is non-linearly realised in the vacuum, with only the electromagnetic subgroup  $U(1) < SU(2) \times U(1)$  being linearly realised. Thus, the electroweak (EW) vacuum is degenerate and the vacua are described by a homogeneous space  $SU(2) \times U(1)/U(1)$ .

The starting point for this Letter is the observation that there are many ways to include  $U(1)$  in  $SU(2) \times U(1)$ ; different ways lead to homogeneous spaces that can be topologically inequivalent. In the Standard Model (SM), the vacuum manifold arises due to a non-vanishing vacuum expectation value (VEV) of the Higgs field, carrying the doublet representation of  $SU(2)$ , and is homeomorphic to the 3-sphere,  $S^3$ . As is well-known, this is rather boring from a physicist's point of view, since the vanishing of the homotopy groups  $\pi_1(S^3)$  (respectively  $\pi_2(S^3)$ ) implies the absence of topologically-stable strings (respectively monopoles). Here, we investigate different inclusions of  $U(1)$ , which lead to vacuum manifolds with fundamental group given by an arbitrary cyclic group; alternatives to the SM based on such inclusions thus feature topologically-stable strings, with potentially interesting consequences for astrophysics, cosmology, and particle physics.

Given the recent discovery [1] of a particle whose properties correspond rather closely to that of the Higgs boson, consideration of alternatives to the SM requires a willing suspension of disbelief on the part of the reader, but there are nevertheless plausible reasons for doing so. Firstly, as we shall see in §VI, there exist models in which the EW hierarchy problem is solved by compositeness of

the Higgs, in which the vacuum manifold is homeomorphic to  $\mathbb{R}P^3$ , with  $\pi_1(\mathbb{R}P^3) = \mathbb{Z}/2$ . Secondly, it is not inconceivable that the Higgs boson is, in fact, an impostor [2]. Thirdly, we shall see that low-energy effective field theory (EFT) descriptions based on topologically non-trivial alternatives, whilst neither as predictive nor as compelling as the SM, can nevertheless reproduce all existing data. Moreover, since they include the SM as a special case, they can be regarded as a consistent way in which to parameterise deformations from the SM and thus provide a meaningful framework for interpreting the results of precision tests of the EW sector. Compared to similar existing frameworks [3], they feature an additional, integral parameter associated with the global topology and it is of interest to speculate as to how this parameter might be constrained using astro- or collider-physics experiments. Finally, even if the Higgs boson turns out to be the real McCoy, the existence of such topologically-distinct alternatives to the status quo is surely a noteworthy theoretical curiosity in its own right.

## II. TOPOLOGY OF $SU(2) \times U(1)/U(1)$

We begin our discussion by assuming that the EW gauge group really is  $G = SU(2) \times U(1)$ , deferring discussion of groups locally isomorphic thereto until the end. We write elements of  $G$  as  $(U, z)$ , where  $U$  is a  $2 \times 2$  unitary matrix with unit determinant and  $z$  is a unit complex number. For  $p, q \in \mathbb{Z}$  there is a homomorphism  $\phi_{p,q} : U(1) \rightarrow G$  given by  $\phi_{p,q}(z) = (\text{diag}(z^q, z^{-q}), z^p)$ , and if  $(p, q)$  are coprime then  $\phi_{p,q}$  is injective, in which case we write  $H_{p,q} \leq G$  for its image. (Any injective homomorphism  $\phi : U(1) \rightarrow G$  is conjugate to some  $\phi_{p,q}$ , as its projection to the  $SU(2)$ -factor may be conjugated to land in the standard maximal torus.)

Our first goal is to investigate the topology of the homogeneous spaces  $G/H_{p,q}$ . An immediate result is that  $G/H_{p,q}$  cannot be homeomorphic for different  $p$ , because

\* gripaios@hep.phy.cam.ac.uk

† o.randal-williams@dpmms.cam.ac.uk

a loop wound once around  $H_{p,q}$  is wound  $p$  times around the  $U(1)$  factor of  $G$ . This implies, using the long exact sequence of homotopy groups  $\pi_1(H_{p,q}) \cong \mathbb{Z} \rightarrow \pi_1(G) \cong \mathbb{Z} \rightarrow \pi_1(G/H_{p,q}) \rightarrow \pi_0(H_{p,q}) \cong 0$  of the fibre bundle  $H_{p,q} \hookrightarrow G \rightarrow G/H_{p,q}$ , that  $\pi_1(G/H_{p,q}) \cong \mathbb{Z}/p$ . Moreover, we see that topologically-stable string configurations occur when  $p \neq 1$  [4].

To investigate the topology further, let  $K_{p,q} = H_{p,q} \cap (SU(2) \times \{1\}) \leq SU(2)$  and consider the function  $\pi : A \mapsto (A, 1)H_{p,q} : SU(2) \rightarrow G/H_{p,q}$ . This is a composition of smooth maps  $SU(2) \hookrightarrow G \rightarrow G/H_{p,q}$  and so smooth. The differential at the identity  $D\pi : \mathfrak{su}(2) \rightarrow \mathfrak{g}/\mathfrak{h}_{p,q}$  is an isomorphism, and by homogeneity it follows that  $\pi$  is a submersion, and hence a local diffeomorphism. Furthermore the right  $K_{p,q}$ -action on  $SU(2)$  acts freely and transitively on the fibres of  $\pi$ , exhibiting it as a principal  $K_{p,q}$ -bundle, and hence giving a diffeomorphism  $SU(2)/K_{p,q} \cong G/H_{p,q}$ .

Now  $K_{p,q} = \{\text{diag}(e^{2\pi i q k/p}, e^{-2\pi i q k/p}) : k \in \mathbb{Z}\}$  is the same subgroup of  $SU(2)$  as  $K_{p,1}$ , because  $(p, q)$  are coprime, and as  $K_{-p,1}$ : thus we shall suppose  $p > 0$ . It follows that  $G/H_{p,q}$  is diffeomorphic to  $SU(2)/K_{p,1}$ , which is further diffeomorphic, as we now show, to a lens space [5]. These spaces are of great historical importance in mathematics, providing the first examples of manifolds whose homeomorphism type is determined by neither their fundamental group and homology [6], nor even their homotopy type [7]. The lens space  $L(n, m)$  is defined for  $(n, m)$  coprime as the quotient of the unit sphere,  $S^3 \subset \mathbb{C}^2$  by the free  $\mathbb{Z}/n$ -action generated by  $(z_1, z_2) \mapsto (e^{2\pi i/n} z_1, e^{2\pi i m/n} z_2)$ . Identifying  $SU(2)$  with the unit sphere  $S^3 \subset \mathbb{C}^2$ ,  $SU(2)/K_{p,1}$  is thus identified with the lens space  $L(p, 1)$ .

The lens spaces  $L(p, 1)$  are precisely those 3-manifolds that arise as principal  $U(1)$ -bundles over the 2-sphere (except for  $S^2 \times U(1)$ ). Indeed, the clutching construction shows that such bundles are in bijection with  $\pi_1(U(1)) = \mathbb{Z}$ , and this bijection may be given by assigning to a principal  $U(1)$ -bundle over the 2-sphere its Euler number.

Writing  $U(1) = \{\text{diag}(e^{i\theta}, e^{-i\theta}) : \theta \in [0, 2\pi)\} \leq SU(2)$ , the Hopf bundle  $h_1 : SU(2) \rightarrow SU(2)/U(1) = S^2$  is the principal  $U(1)$ -bundle with Euler number 1. As  $K_{p,1} \leq U(1)$  the map  $h_1$  is the composition

$$SU(2) \longrightarrow SU(2)/K_{p,1} \xrightarrow{h_p} SU(2)/U(1) = S^2$$

of a  $p$ -fold covering map and a principal  $(U(1)/K_{p,1} \cong U(1))$ -bundle  $h_p$ , whose Euler number is therefore  $p$  and whose total space is  $G/H_{p,q} \cong SU(2)/K_{p,1} \cong L(p, 1)$ .

From this perspective, we may use standard results to read off the algebraic topological invariants of  $G/H_{p,q}$ : the homotopy groups are given by  $\pi_1 = \mathbb{Z}/p, \pi_{i>1} = \pi_{i>1}(S^3)$  (so  $\pi_2 = 0, \pi_3 = \mathbb{Z}, \pi_4 = \mathbb{Z}/2, \&c.$ ); the integral cohomology is given by  $H^0 = \mathbb{Z}, H^1 = 0, H^2 = \mathbb{Z}/p, H^3 = \mathbb{Z}$ . Most interesting among these, for physicists, is  $\pi_1 = \mathbb{Z}/p$ .

How do these results relate to the SM? In that case, we postulate the existence of a Higgs field, that is a matter

field  $\phi$  whose potential is such that it acquires a non-vanishing VEV. It carries the doublet irreducible representation of  $SU(2)$  and its charge  $q \in \mathbb{Z}$  under  $U(1)$  is non-vanishing, but otherwise arbitrary. The  $G$ -action is then  $G : \phi \mapsto U z^q \phi$ . Without loss of generality, we may write the Higgs VEV as  $\langle \phi \rangle = (0, v)^T$ , such that the unbroken subgroup is  $H_{1,q} = \{(\text{diag}(z^q, z^{-q}), z)\}$ . The discussion above then shows that, as expected, the SM EW vacuum manifold is homeomorphic to  $S^3$  and does not feature topologically-stable strings.

### III. NON-LINEAR SIGMA MODEL

Having established the existence of homogeneous spaces with non-trivial topology, we now construct physical theories based upon them, beginning with non-linear sigma models (NLSMs). These represent the most general low energy-effective field theories consistent with the non-linearly realised symmetry  $G$  [8] and we would like to show that such theories can also be consistent with experimental data for any  $p, q$ .

To do so is a triviality, at least insofar as particle physics experiments are concerned. Indeed, even in the ungauged theory, physics which depends only on local properties of the vacuum manifold can only depend on  $p$  and  $q$  through their quotient. But even the quotient is unphysical (locally) in the gauged theory, because differing values of  $p$  and  $q$  can be absorbed by redefinitions of the gauge coupling constants.

Thus, locally, all such models are equivalent to models with  $p = 1$ . But models with  $p = 1$  (with the Higgs boson treated as an additional,  $CP$ -even, singlet, scalar matter field with arbitrary couplings [3]) contain the SM (or rather a low-energy limit thereof) as a special case and thus can be consistent with data. So for any  $p$ , there exists a choice of parameters in the corresponding NLSM that is consistent with all particle physics data.

The question of whether such a model is consistent with astrophysical data is rather harder to settle. The model would feature topologically-stable string solutions and one would expect a network of these to form in the early Universe during the cosmological EW phase transition [4]. The purely gravitational effects of such strings are of order  $v^2/M_P^2 \sim 10^{-34}$  and are utterly negligible. But such a string features quark and lepton zero modes localised on its core, which lead to the formation of superconducting currents (as well as baryon- and lepton-number violation) in the presence of astrophysical magnetic fields [9]. A number of resulting astrophysical signatures have been discussed (in, among others, the microwave background, radio bursts, cosmic rays, and galactic and stellar dynamics; for a review, see [10]) but there seems to be no consensus that they lead to robust constraints on EW-scale strings.

It would also be of interest to study the production and subsequent decay of loops of such strings at multi-TeV colliders, such as the LHC [11].

#### IV. CUSTODIAL SYMMETRIES

Though a NLSM based on  $G/H_{p,q}$  can fit the data for any  $p$ , it is clearly far less satisfactory than the SM as regards its predictivity. Most seriously, while the Higgs boson is an integral part of the SM, in an NLSM description we are forced to arbitrarily include an additional scalar matter field whose couplings are tuned to be close to those of the SM Higgs boson. Even if we are willing to overlook these issues regarding the Higgs, there are two more successful predictions of the SM which, though they can be accommodated by any model, are not predicted in a generic NLSM. The first of these is the  $W - Z$  boson mass ratio, which is fixed in the SM, but is arbitrary in the  $G/H_{p,q}$  NLSM. Indeed, the gauge boson masses are determined by specifying a  $G$ -invariant metric on  $G/H_{p,q}$ . In the SM, the metric is fixed to be the round metric on  $S^3$ , which is unique up to an overall normalization, so that the  $W - Z$  mass ratio is fixed. But in a generic NLSM, we may pick an arbitrary  $G$ -invariant metric on  $G/H_{p,q}$ . Such metrics may be classified as follows [12]: as  $G$  acts almost effectively on  $G/H_{p,q}$  (i.e. the subgroup of  $G$  that fixes all elements of  $G/H_{p,q}$  is discrete), the  $G$ -invariant metrics on  $G/H_{p,q}$  are in 1-1 correspondence with those inner products on  $\mathfrak{g}/\mathfrak{h}_{p,q}$  which are invariant for the adjoint action of  $H_{p,q}$ . The adjoint representation of  $SU(2) \times U(1)$  restricts to  $H_{p,q}$  as  $3 \oplus 1 \rightarrow 2q \oplus -2q \oplus 0 \oplus 0$ , where we denote representations on the LHS by their dimension and on the RHS by their  $H_{p,q}$  charge. There are thus 2 such independent inner products, each with an arbitrary overall normalization, leading to an arbitrary  $W - Z$  mass ratio.

As is well-known, the particular mass ratio that obtains in the SM can be understood via custodial symmetry [13]: the vacuum manifold  $S^3$  is invariant under a larger action of  $SU(2) \times SU(2)$  (with the original  $U(1)$  included in the second  $SU(2)$ ), broken to the diagonal  $SU(2)$  by the Higgs VEV. Since the adjoint representation of  $SU(2) \times SU(2)$  restricts as  $(3, 1) \oplus (1, 3) \rightarrow 3 \oplus 3$ , there is just one invariant inner product, up to an overall normalization.

We claim that a similar construction can be applied to the  $G/H_{p,q}$  NLSM only in the cases  $p \in \{1, 2\}$ . Indeed, to do so requires us to find a metric on  $G/H_{p,q}$  which is invariant not just under  $SU(2) \times U(1)$ , but rather under the larger  $SU(2) \times SU(2)$ . Now, this larger group need not act effectively on  $G/H_{p,q}$ , but the subgroup  $N = \{g \in SU(2) \times SU(2) : g.x = x \ \forall x \in G/H_{p,q}\}$ , being the kernel of a homomorphism from  $SU(2) \times SU(2)$  to  $\text{Sym}(G/H_{p,q})$ , will necessarily be normal, and furthermore  $(SU(2) \times SU(2))/N$  will act effectively on  $G/H_{p,q}$ . Now, the normal subgroups of  $SU(2)$  are  $\{e\}, \mathbb{Z}/2, SU(2)$  and projecting  $N$  to either  $SU(2)$ -factor must give one of these. But neither projection can be  $SU(2)$ , as otherwise  $(SU(2) \times SU(2))/N$  would have dimension at most 3, which is incompatible with the known restriction of the action to the subgroup  $SU(2) \times U(1)$  (in which both factors have a non-trivial

action), so in fact  $N$  must be a subgroup of the centre  $\mathbb{Z}/2 \times \mathbb{Z}/2$ . Thus the  $SU(2) \times SU(2)$ -action is almost effective and the isometry group of the desired metric has dimension at least 6. But it is a theorem [14] that the isometry group of a compact Riemannian 3-manifold has dimension at most 6, equalling 6 only if it is  $S^3$  or  $\mathbb{R}P^3$ , corresponding to  $p = 1$  or  $p = 2$  respectively.

Thus, we see that a custodial symmetry protecting  $W$  and  $Z$  masses may also be imposed in the topologically non-trivial case  $p = 2$ . Here  $G/H_{2,1} \cong \mathbb{R}P^3 \cong SO(3)$  has an action of  $SO(3) \times SO(3)$  by  $(g, h).x = gxh^{-1}$ , with the  $SU(2) \times SU(2)$ -action being the one induced by the covering map from  $SU(2) \rightarrow SO(3)$ ; the LSM model (respectively composite Higgs model) in § V (respectively § VI) give explicit realisations.

The second successful prediction of the SM involves the couplings of gauge bosons to fermions. Again, for a generic  $G/H_{p,q}$  NLSM, deviations in these couplings may occur. In a concrete theory of flavour such as partial compositeness (which has the *desiderata* of explaining much of the hierarchical structure of Yukawa couplings whilst suppressing potential flavour-changing effects) [15] this is not so much of a problem, because the SM fermions are largely elementary, being weakly mixed with the strong sector. Deviations are therefore expected to be small. The biggest problem arises in the coupling of the  $Z$  boson to left-handed bottom quarks, since the latter belongs to the same  $SU(2)$  doublet as the top quark, and is forced to be sizably mixed with the sigma model sector in order to accommodate the large top quark mass. But this too can be protected by a symmetry in the  $p = 1$  case [16]. The required group is  $(SU(2) \times SU(2)) \rtimes \mathbb{Z}/2$ , where  $\mathbb{Z}/2$  permutes the two  $SU(2)$ s [17]. A similar protection can also be achieved when  $p = 2$ , by using the action of  $(SO(3) \times SO(3)) \rtimes \mathbb{Z}/2$  on  $G/H_{2,1} \cong \mathbb{R}P^3 \cong SO(3)$ , where the  $\mathbb{Z}/2$  acts by inversion on the group  $SO(3)$ .

#### V. LINEAR SIGMA MODELS

We may also be more ambitious and try to construct a UV-complete model with a non-trivial EW vacuum manifold. A model in which the Higgs field of the SM is replaced by a scalar carrying the  $(2j+1, q)$  representation of  $SU(2) \times U(1)$  (where the spin  $j$  representation of  $SU(2)$  is labelled by its dimension  $2j+1$ ) leads to a vacuum manifold homeomorphic to  $L(p, 1)$ , with  $p = 2j/\text{gcd}(2j, q)$ . Either the arguments given in §III or an explicit calculation shows that the pattern of couplings of gauge bosons to themselves and to fermions is exactly as in the SM, but, since custodial symmetry is not respected, there is a gross violation of the  $W - Z$  mass ratio for  $2j \neq 1$ , with  $\frac{m_W^2}{m_Z^2} = \frac{g_2^2}{2j(g_2^2 + g_1^2)}$  at tree-level, where  $g_2$  (respectively  $g_1$ ) denotes the SM value of the  $SU(2)$  (resp.  $U(1)$ ) gauge coupling.

For a model yielding the correct tree-level value of  $W - Z$  mass ratio, we replace the SM Higgs field by a real

scalar,  $\Phi$ , carrying a bi-triplet,  $(3, 3)$ , representation of  $O(3) \times O(3)$ . We employ a matrix notation where the action of  $(L, R) \in O(3) \times O(3)$  is given by  $\Phi \mapsto L\Phi R^T$ . We may then define an action of  $SU(2) \times SU(2)$  on  $\Phi$  via the usual covering map  $\pi : SU(2) \rightarrow SO(3) < O(3)$  and so there is also an action of the SM electroweak gauge group  $SU(2) \times U(1)$  (with  $U(1)$  a subgroup of the other  $SU(2)$ ).

The most general, renormalizable,  $O(3) \times O(3)$ -invariant potential for  $\Phi$  is a linear combination of  $\text{tr}\Phi^T\Phi$ ,  $\text{tr}(\Phi^T\Phi)^2$ , and  $(\text{tr}\Phi^T\Phi)^2$ , which may be written in the form  $V(\Phi) = a(\text{tr}\Phi^T\Phi - c^2)^2 + b(\text{tr}\Phi^T\Phi - 3c^2)^2$ , where  $a, b, c^2 \in \mathbb{R}$  and  $a, b > 0$ . Written thus, it is clear that the minimum of the potential lies at  $\Phi^T\Phi = c^2 1$  (for  $c^2 > 0$ ). Not only does this tell us that the unbroken subgroup of  $O(3) \times O(3)$  is indeed the diagonal  $O(3)$  (since we can choose  $\langle \Phi \rangle = c1$  such that  $\langle \Phi \rangle \mapsto cLR^T$  which equals  $\langle \Phi \rangle = c1$  iff.  $L = R$ ), but it also tells us that the vacuum manifold is homeomorphic to  $O(3)$ , whose component connected to the identity is homeomorphic to  $SO(3) \cong \mathbb{R}P^3$ . Since  $(3, 3)$  restricts to  $1 \oplus 3 \oplus 5$  under  $O(3) < O(3) \times O(3)$ , we expect to find a spectrum in the Higgs sector consisting of the 3 massless Goldstone bosons, together with a further 6 massive states, of which at least 5 are degenerate in mass. Indeed, if we expand  $V(\Phi)$  to quadratic order using  $\Phi = (c+h)1 + S + A$ , where  $S$  is traceless and symmetric and  $A$  is antisymmetric, we find that the first term in the potential generates equal masses for  $S$  and  $h$ , while the second term (because it is, in fact,  $O(9)$  symmetric) generates a mass for  $h$  only.

This particular model is something of a curate's egg, from the phenomenological point of view. On the one hand, custodial symmetry implies that we obtain  $W$  and  $Z$  boson masses consistent with observations, together with a massive custodial singlet  $h$  that resembles the SM Higgs boson in its couplings to gauge bosons. On the other hand, the quintuplet mass  $S$  cannot exceed the Higgs mass, which may well lead to conflict with searches for charged Higgs bosons. More worryingly, there is no way to generate acceptable masses for SM fermions in this model via Yukawa couplings, since  $\Phi$  can only couple to pairs of  $SU(2)_L$  doublets or pairs of singlets, rather than to a doublet and a singlet as in the SM [18].

## VI. A COMPOSITE HIGGS MODEL

It is also possible to achieve a topologically non-trivial vacuum manifold in models in which the hierarchy problem is solved by making the Higgs boson a composite of some new TeV-scale dynamics. Indeed, in the most favoured among such models [16], the Higgs is a pseudo-Goldstone boson, taking values in a homogeneous space  $SO(5)/O(4) \simeq \mathbb{R}P^4$ . Due to the presence of couplings

to gauge bosons and fermions, the dynamics cannot be fully  $SO(5)$  invariant and so the low-energy effective lagrangian contains a potential for the Higgs field. This is a real-valued,  $SU(2) \times U(1)$ -invariant function on  $\mathbb{R}P^4$ , with the vacuum manifold being given by the level set of the minimum. The specific form of the function is determined by the uncalculable strong dynamics, but let us suppose, for the sake of illustration, that it takes the form  $V = \frac{x_1^2}{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2}$ , where  $x_i, i \in \{1, 2, 3, 4, 5\}$  are coordinates in  $\mathbb{R}^5$ . This is well-defined on  $\mathbb{R}P^4$ , the space of lines through the origin in  $\mathbb{R}^5$ , and is, moreover, smooth and invariant under the larger group  $O(4)$ . We have that  $0 \leq V \leq 1$ , with the maximum at the point  $x_2 = x_3 = x_4 = 0$  and the minimum at  $x_1 = 0$ , corresponding to the level set  $\mathbb{R}P^3$ . While our ignorance of strong dynamics makes it hard to determine whether such a model is excluded or not, order of magnitude estimates indicate that a vacuum with a rather smaller value of  $x_1 \lesssim 0.1$  (in which case the low-energy physics approximates that of the SM) is preferred.

## VII. GROUPS LOCALLY ISOMORPHIC TO $SU(2) \times U(1)$

Finally, we consider gauge groups that are locally, but not globally, isomorphic to  $SU(2) \times U(1)$ . Such groups need not be connected, in which case the possibilities are infinite, but all feature domain-wall type solutions that are potentially dangerous from a cosmological point of view. If we restrict our attention to the component connected to the identity, then the possibilities are finite in number, given by quotients of the universal cover  $SU(2) \times \mathbb{R}$  by a discrete subgroup of the centre  $\mathbb{Z}/2 \times \mathbb{R}$ . Of the five possibilities, the only ones other than  $SU(2) \times U(1)$  admitting doublet irreducible representations (as carried by quarks and leptons) are  $SU(2) \times \mathbb{R}$  and  $U(2)$ . The former has subgroups isomorphic to  $\mathbb{R}$  and is disfavoured by the apparent quantization of hypercharge; similar arguments to those given in §II show that the vacuum manifold is always homeomorphic to  $S^3$  in this case. The latter has  $U(1)$  subgroups given by  $H_{r,s} = \{\text{diag}(e^{ir\theta}, e^{is\theta})\}$ , with  $r, s \in \mathbb{Z}$  and coprime, is homeomorphic to  $L(r+s, 1)$ , and also admits topologically-stable strings.

## ACKNOWLEDGEMENTS

BG acknowledges the support of STFC grant ST/L000385/1 and both authors acknowledge the support of King's College, Cambridge.

---

[1] G. Aad *et al.* (ATLAS), Phys. Lett. **B716**, 1 (2012), arXiv:1207.7214 [hep-ex]; S. Chatrchyan *et al.* (CMS),

Phys. Lett. **B716**, 30 (2012), arXiv:1207.7235 [hep-ex].

- [2] See, *e.g.*, W. D. Goldberger, B. Grinstein, and W. Skiba, Phys. Rev. Lett. **100**, 111802 (2008), arXiv:0708.1463 [hep-ph].
- [3] R. Contino, C. Grojean, M. Moretti, F. Piccinini, and R. Rattazzi, JHEP **05**, 089 (2010), arXiv:1002.1011 [hep-ph].
- [4] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, 2000).
- [5] H. F. F. Tietze, Monatshefte für Mathematik und Physik **19**, 1 (1908).
- [6] J. W. Alexander, Trans. Amer. Math. Soc. **20**, 339 (1919).
- [7] W. Threlfall and H. Seifert, Math. Ann. **107**, 543 (1933); K. Reidemeister, Abh. Math. Semin. Hamb. Univ. **11**, 102 (1935); J. H. C. Whitehead, Ann. of Math. **42**, 1197 (1941); E. J. Brody, *ibid.* **71**, 163 (1960).
- [8] S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. G. Callan, Jr., S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969).
- [9] E. Witten, Nucl. Phys. **B249**, 557 (1985).
- [10] M. B. Hindmarsh and T. W. B. Kibble, Rept. Prog. Phys. **58**, 477 (1995), arXiv:hep-ph/9411342 [hep-ph].
- [11] *cf.* Y. Nambu, Nucl. Phys. **B130**, 505 (1977).
- [12] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. 2 (John Wiley & Sons, 1969).
- [13] P. Sikivie, L. Susskind, M. B. Voloshin, and V. I. Zakharov, Nucl. Phys. **B173**, 189 (1980).
- [14] S. Kobayashi, *Transformation groups in differential geometry* (Springer-Verlag, 1972).
- [15] D. B. Kaplan, Nucl. Phys. **B365**, 259 (1991).
- [16] K. Agashe, R. Contino, L. Da Rold, and A. Pomarol, Phys. Lett. **B641**, 62 (2006), arXiv:hep-ph/0605341 [hep-ph].
- [17] B. Gripaios, T. Müller, M. A. Parker, and D. Sutherland, JHEP **08**, 171 (2014), arXiv:1406.5957 [hep-ph].
- [18] If one reinstates the SM Higgs to fix this, as in the model of [19], the vacuum manifold is restored to  $S^3$ .
- [19] H. Georgi and M. Machacek, Nucl. Phys. **B262**, 463 (1985); M. S. Chanowitz and M. Golden, Phys. Lett. **B165**, 105 (1985).